



## Some Basic Diophantine Equations

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**Abstract-** In this paper we present a method for solving the Diophantine equation, first we find the polynomial solution for the PELL'S equation by the method of continued fractions then present the integral solution of the Diophantine equations. Theorems are discussed to demonstrate the continue fraction method.

**Keywords-** Diophantine equations, PELL'S equations and Continue fraction method.

### I. INTRODUCTION

An equation which has two or more than two unknowns is called an indeterminate equation. More generally, a system of equations is called indeterminate if the number of equations is less than that of the unknowns. Diophantus, one of the last Alexandrian Mathematicians of 3<sup>rd</sup> century devoted himself to algebra. He proposed many indeterminate problems in this Arithmetic. He was content with a single numerical rational solution, although the problems usually have infinitely many rational solution and often integral solutions. Because he some times restricted his solutions to integer, in his honour, his name is attached to the kind of indeterminate equations for which the values of the variables are integers and study of such equation is known as DIOPHANTINE ANALYSIS. One can easily understand that Diophantine problems offer unlimited field for research by reason of their variety.

This dissertation consists of three chapters.

In chapter 1, we present of polynomial solutions for the PELL'S EQUATIONS  $A^2 - DB^2 = \pm 1$  by employing the method of continued fractions.

In chapter 2, we present integral solutions of the Diophantine Equations

$$i) \quad A^{nh} = B^h + C^h + D^h \quad (h = 1, 2)$$

$$ii) \quad A^{nh} + A^{nh} = B^{2h} + C^{2h} + D^{2h} \quad (h = 1, 2)$$

In chapter 3, we are consider the famous Fermat's theorem and present an elegant way of proving the above theorem.

### II. POLYNOMIAL SOLUTIONS FOR THE PELL'S EQUATION $A^2 - DB^2 = \pm 1$ INTRODUCTION

Let D be a given square free natural number. It is well known that the Pell's equations  $A^2 - DB^2 = 1$  .... (1) always has a infinite number of integral solutions of (1) with positive A and B are obtained by the formula  $A^r + B^r D = (a + b\sqrt{D})^r$  where  $r = 1, 2, 3, \dots$  and  $a + b\sqrt{D}$  is the fundamental solution of (1)  $A_1 = a$  and  $B_1 = b$ .

The equations  $A^2 - DB^2 = -1$  ..... (2) may not always have integral solutions. When this equation is solvable in integer for some given D, if  $L + M\sqrt{D}$  be its fundamental solutions.

Therefore we have the relation  $a + b\sqrt{D} = (L + M\sqrt{D})^2$  ..... (3) where  $a + b\sqrt{D}$  is the fundamental solution of (1).

For a given value of D, in order to determine the fundamental solution of (1) or (2) (when it exists). One may use the method of continued fraction expansion of  $\sqrt{D}$ . The penultimate convergent of this expansion provides the fundamental solutions. From computational point of view, it would be advantageous to group value of D into several classes such that the values of D in a specific class are associated with a similar pattern of continued fraction expansions. It is found that polynomial expressions for D come in handy for this purpose. As a consequence, for D some polynomial solutions are given in this paper, for the Pell's equations  $A^2 - DB^2 = \pm 1$ .

### FOR THE PELL'S EQUATION $A^2 - DB^2 = 1$

Theorem 1

Suppose  $D = 4t^2 + 12t + 5$  where  $t$  is any natural number then  $a + b\sqrt{D}$  is the fundamental solution of  $A^2 - DB^2 = \pm 1$ .

Proof:

For real  $x$ , let the symbol  $[x]$  denote the greatest integer  $\leq x$ . We have  $(2t + 2) < \sqrt{D} < (2t + 3)$  and so  $[D] = 2t + 2$ . Now,



expanding D into a continued fraction expansion, we obtain

$$\begin{aligned}\sqrt{D} &= [\sqrt{D}] + \sqrt{4t^2 + 12t + 5} - (2t + 2) \\ &= (2t + 2) + \frac{4t + 1}{\sqrt{4t^2 + 12t + 5} + (2t + 2)} \\ &= (2t + 2) + \frac{1}{\frac{\sqrt{4t^2 + 12t + 5} + (2t + 2)}{4t + 1}} \\ &= (2t + 2) + \frac{1}{\frac{\sqrt{4t^2 + 12t + 5} - (2t - 1)}{4t + 1}} \\ &= (2t + 2) + \frac{1}{1 + \frac{1}{t + \frac{8t + 4}{4[\sqrt{4t^2 + 12t + 5} + (2t + 1)]}}} \\ &= \dots \\ &= |2t + 2, 1, t, 2, t, 1, 4t + 4|\end{aligned}$$

The penultimate convergent is

$$\begin{aligned}&2t + 2 + \frac{1}{1 + \frac{1}{t + \frac{1}{2 + \frac{1}{t + 1}}}} \\ &= 2t + 2 + \frac{1}{1 + \frac{1}{t + \frac{t + 1}{2t + 3}}} \\ &= 2t + 2 + \frac{1}{1 + \frac{2t + 3}{2t^2 + 4t + 1}} \\ &= 2t + 2 + \frac{2t^2 + 4t + 1}{2t^2 + 6t + 4} \\ &= \frac{4t^3 + 18t^2 + 29t + 9}{2t^2 + 6t + 4}\end{aligned}$$

Thus we get the relation

$$(4t^3 + 18t^2 + 29t + 9)^2 - (4t^2 + 12t + 5)(2t^2 + 6t + 4)^2 = 1$$

### Theorem 2:

Suppose  $D = m^2t^4 + 2mnt^3 + n^2t^2 + mt + n$  where m, n, t are any natural numbers with  $mn \neq 0$ .

Then  $a + b\sqrt{D}$  is the fundamental solutions of  $A^2 - DB^2 = 1$  where

$$a = 2mt^3 + 2nt^2 + 1, b = 2t.$$

### Proof:

For real x, let the symbol  $[x]$  denote the greatest integer  $\leq x$ . We have  $(mt^2 + nt) < D < (mt^2 + nt)^2$  and

so  $[\sqrt{D}] = mt^2 + nt$ . Now, expanding  $\sqrt{D}$  in to a continued fraction expansion.

$$\begin{aligned}[\sqrt{D}] &= (mt^2 + nt) + \sqrt{m^2t^4 + 2mnt^3 + n^2t^2 + mt + n} - (mt^2 + nt) \\ &= (mt^2 + nt) + \frac{1}{\frac{\sqrt{m^2t^4 + 2mnt^3 + n^2t^2 + mt + n}}{(mt^2 + nt)}} \\ &= (mt^2 + nt) + \frac{1}{\frac{2t + \sqrt{m^2t^4 + 2mnt^3 + n^2t^2 + mt + n}}{mt + n}} \\ &= \dots \\ &= |mt^2 + nt, 2t, 2mt^2 + 2nt|\end{aligned}$$

The penultimate convergent is

$$\begin{aligned}&mt^2 + nt + \frac{1}{2t} \\ &= \frac{2mt^3 + 2nt^2 + 1}{2t}\end{aligned}$$

Thus we get the relation

$$(2mt^3 + 2nt^2 + 1)^2 - (m^2t^4 + 2mnt^3 + 2nt^2 + mt + n)(2t)^2 = 1$$

### Theorem 3:

Suppose  $D = m^2t^4 + 2mnt^3 + n^2t^2 + 2mt + 2n$  where m, n, t are any natural numbers with  $mn \neq 0$ .

Then  $a + b\sqrt{D}$  is the fundamental solution of  $A^2 - DB^2 = 1$  where  $a = mt^3 + nt^2 + 1$ ,  $b = t$ .

Proof :

Consider  $[D] = mt^2 + nt$  and do the proof by the continued fraction doing the above 2 theorem, we get the expansion of

$$\begin{aligned}\sqrt{D} &= |mt^2 + nt, t, 2mt + 2nt| \\ \text{The penultimate convergent is } &\frac{mt^3 + nt^2 + 1}{t}\end{aligned}$$

$$\begin{aligned}\text{Thus we get the relation } &(mt^3 + nt^2 + 1)^2 \\ &- (m^2t^4 + 2mnt^3 + n^2t^2 + 2mt + 2n)(t^2) = 1\end{aligned}$$

FOR THE PELL'S EQUATION  $A^2 - DB^2 = -1$

### Theorem 4:

If  $D = 4t^2 + 4t + 5$  where t is any natural number then  $L + M\sqrt{D}$  is the fundamental solution of

$$\begin{aligned}A^2 - DB^2 &= -1 \text{ where } L = 4t^3 + 6t^2 + 6t + 2, \\ m &= 2t^2 + 2t + 1\end{aligned}$$

Proof:



For real  $x$ , let the symbol  $[x]$  denote the greatest integer  $\leq x$ . We have  $(2t+1) < \sqrt{D} < (2t+2)$ .. and so  $[\sqrt{D}] = 2t+1$  Now, expanding  $\sqrt{D}$  into a continued freedom fraction expansion.

We obtain

$$\begin{aligned}\sqrt{D} &= [D] + \sqrt{4t^2 + 4t + 5} - (2t + 1) \\ &= (2t + 1) + \frac{1}{t + \frac{1}{\sqrt{4t^2 + 4t + 5} + 2t}} \\ &= (2t + 1) + \frac{1}{t + \frac{1}{1 + \frac{1}{\sqrt{4t^2 + 4t + 5} + (2t - 1)}}}\end{aligned}$$

= .....

=  $[2t+1, t, 1, 1, t, 4t+2]$

The penultimate convergent is

$$\begin{aligned}2t+1 + \frac{1}{t + \frac{1}{1 + \frac{1}{1 + \frac{1}{t+1}}}} \\ = \frac{4t^3 + 6t^2 + 6t + 2}{2t^2 + 2t + 1}\end{aligned}$$

Thus we get the solution

$$= (4t^3 + 6t^2 + 6t + 2)^2 - (4t^2 + 4t + 5)(2t^2 + 2t + 1)^2 = -1$$

### Theorem 5:

If  $D = 16t^4 + 8t^3 + 9t^2 + 6t + 2$  where  $t$  is any natural number then  $L + M\sqrt{D}$  is the fundamental solution of  $A^2 - DB^2 = -1$  where

$$L = 16t^4 + 4t^3 + 8t^2 + 3t + 1, \quad m = 4t^2 + 1$$

**Proof:**

Consider  $[\sqrt{D}] = 4t^2 + t + 1$  and do the proof by the expansion of  $\sqrt{D}$  in a continued fraction we obtain  $[4t^2 + t + 1, 2t, 2t, 8t^2 + 2t + 2]$

The penultimate convergent is  $\frac{16t^4 + 4t^3 + 8t^2 + 3t + 1}{4t^2 + 1}$

Thus we get the relation

$$(16t^4 + 4t^3 + 8t^2 + 3t + 1)^2 - (16t^4 + 8t^3 + 9t^2 + 6t + 2)(4t^2 + 1)^2 = -1$$

### Theorem 6:

If  $D = 4t^2 + 4t + 2$ , where  $t$  is any natural number  $\geq 2$ , then  $L + M\sqrt{D}$  is the fundamental

solution of  $A^2 - DB^2 = -1$  where  $L = 4t^4 - 4t^3 + 2t^2 - 1$   $M = 2t^2 - 2t + 1$ .

**Proof:**

Consider  $[\sqrt{D}] = 2t^2$  and do the proof by the expansion of  $\sqrt{D}$  in a continued fraction are obtain  $[2t^2, t-1, 1, 1, t-1, 4t^2]$

The penultimate convergent is  $\frac{4t^4 - 4t^3 + 2t^2 + 2t - 1}{2t^2 - 2t + 1}$

Thus we get the relation

$$(4t^4 - 4t^3 + 2t^2 + 2t - 1)^2 - (4t^4 + 4t + 2)(2t^2 - 2t + 1)^2 = -1$$

## THE GENERAL PELL'S EQUATIONS

### Theorem 7:

Let  $u, v$  be integer such that  $u^2 - Dv^2 = N$ , where  $N$  is any given non zero integer. Then

$$A_{(1)} = Kv^2 \pm u, \quad B_{(1)} = V,$$

$$D_{(1)} = K^2V^2 \pm 2ku + D \quad \text{satisfy}$$

$A_{(1)}^2 - D_{(1)}B_{(1)}^2 = N$ , Where  $K$  is any natural number

**Proof:**

Given

$$A_{(1)} = Kv^2 \pm u$$

$$B_{(1)} = V$$

$D_{(1)} = K^2v^2 \pm 2Ku + D$  where  $K$  is any natural numbers choose for  $K$  any value satisfying  $A_{(1)}^2 - D_{(1)}B_{(1)}^2 = N$  from the help of above theorem, one may generate the solution of the general Pell's equation.

$u^2 - Dv^2 = N$  where  $N$  is a non zero integer

**Corollary :** Let  $K, t$  be natural number then  $A = Kt^2 \pm 1$ ,  $B = t$ ,  $D = K^2t^2 \pm 2k$  satisfy  $A^2 - DB^2 = 1$

**THE DIOPHANTINE EQUATION**  $A^2 - DB^4 = \pm 1$

The solution of the Diophantine equations  $A^2 - DB^4 = \pm 1$  may be generated from the solution of the Pell's equation  $u^2 - 2v^2 = \pm 1$ .

### Theorem 8 :

Let  $u, v$  be integer such that  $u^2 - 2v^2 = \pm 1$ . Then  $A = u(V^k \pm 1)$   $B = V$ ,  $D = u^2 \pm 4$  satisfy  $A^2 - DB^{2k} = \pm 1$

**Proof:**

Given:

$$A = u(v^K \pm 1)$$

$$B = V$$

$$D = u^2 \pm 4$$

Therefore it satisfies  $A^2 - DB^{2k} = \pm 1$ 

$$\begin{aligned} u^2 \pm 2u^2v^k \pm 4V^{2k} \\ = u^2 \pm 2v^k(u^2 - 2v^k) \\ = u^2 \pm 2v^k(\pm 1) \\ = u^2 - 2v^k \\ = \pm 1 \end{aligned}$$

When  $k=2$ , the given conditions  $u^2 - 2vk = \pm 1$  reduces to PELL'S Equation  $u^2 - 2v^2 = \pm 1$  with  $D=2$  and hence if possesses and infinite number of solutions for both signs. Thus there are an infinite number of values of  $D$  for which be equation  $A^2 - DB^4 = \pm 1$  has non- trivial integer solutions, as provided by the given statement. However, when  $k \geq 3$ , by Thue's Theorem (Mordell),  $u^2 - 2vk = \pm 1$  has atmost a finite number of integral solutions. For example in the case of  $k=4$ , Ljunggren proved that the only the integer solution of the Diophantine equation.

$$u^2 - 2v^4 = -1$$

Are  $(u,v) = (1,1)$  and  $(239,13)$ .

Using the better solution of this equation in the given theorem, we obtain the relation,  
 $6826318^2 - 57125(13)^8 = -1$

Thus providing a solution of the Diophantine equation.

$$A^2 - DB^8 = -1 \text{ with } D=57125$$

Also considering finally, knowing the solutions  $(A_0, B_0)$   $A^2 = DB^2 \pm 1$ , the solutions for the equation  $A^2 = DB^2 \pm \sigma^2$  are given by  $(\sigma A_0, \sigma B_0)$ .

**SOLUTION OF DIOPHANTINE EQUATION**

A method of constructing special solution of Diophantine Equations:

$$i) \quad A^{nh} = B^h + c^h + D^h \quad (h=1, 2)$$

$$ii) \quad A^{nh} + A^{nh} = B^{2h} + c^{2h} + D^{2h} \quad (h=1, 2)$$

has been described. We know that every prime  $p \equiv 1 \pmod{6}$  (or) every prime of the form  $6k+1$  can be expressed in the form  $p^2 + pq + q^2 \rightarrow (1)$  where  $p, q$  are +ve integer

Also even integer  $N$  which is the product of primes of the form  $6K+1$  can be represented in the form (1) repeated applications of the formula.

$$(a^2 + ab + b^2)(c^2 + cd + d^2) = x^2 - xy + y^2$$

$$\rightarrow (2)$$

Where  $x=ac+bc+bd$ ,  $y=ac+ad+bd$ 

we therefore observe that the product of 2 primes of the form  $6K+1$  is again a prime of the form  $6K+1$ .

**For example:**

$$7 \equiv 1 \pmod{6}, 19 \equiv 1 \pmod{6}$$

$$7 \times 19 \equiv 133 \equiv 1 \pmod{6}$$

If  $N$  can be expressed in the form (1) the  $N''$ , above  $n$  being natural number  $>1$  can also be expressed in the same form using the identity (2)

Now we consider the Diophantine Equations

$$N^{nh} = x^h + y^h + z^h \quad (h=1, 2) \quad \text{where}$$

$$N = p^2 + pq + q^2 \rightarrow (3)$$

Setting  $x = -pq$  and substituting  $h=1, 2$  we have in turn in equation (3), we obtain respectively,.

$$y + z = N - x = p^2 + pq + q^2 + pq = (p + q)^2$$

$$\rightarrow (4a)$$

$$y^2 + z^2 = N^2 - x^2 = (N + x)(N - x) = (p^2 + q^2)(p + q)^2$$

$$\rightarrow (4b)$$

Consider the identity

$$(y - z)^2 = 2(y^2 + z^2) - (y + z)^2$$

Using equation (4a), (4b) we have

$$\begin{aligned} (y - z)^2 &= 2(p^2 + q^2)(p + q)^2 - (p + q)^4 \\ &= (p + q)^2 \{ 2(p^2 + q^2) - (p + q)^2 \} \\ &= (p + q)^2 [p^2 + q^2 - 2pq] \\ &= (p + q)^2 (p - q)^2 \end{aligned}$$

$$(y - z)^2 = (p^2 - q^2)^2 \rightarrow (5)$$

Taking +ve square root

$$(y - z) = (p^2 - q^2) \rightarrow (5a)$$

From (4a) &amp; (5a) we obtain

$$y = p^2 + pq, \quad z = q^2 + pq$$

Substituting the values of  $x, y, z$  in (3) we obtain the identities

$$\left. \begin{aligned} (p^2 + pq + q^2) &\equiv (-pq) + (p^2 + pq) + (q^2 + pq) \\ (p^2 + pq + q^2) &\equiv (-pq)^2 + (p^2 + pq)^2 + (q^2 + pq)^2 \end{aligned} \right\} \rightarrow (6)$$

Note that we obtain the same equation (6) when we substitute the -ve square root in equation (5)



If  $N = C(p^2 + pq + q^2)$  where  $C$  has only prime factors of the form  $6K+5$  or  $2$ , we multiply the system (3) by  $C$  and obtain the same identities (6)

Also then

$$N^n = C^n (p^2 + pq + q^2)^n, n > 1$$

$$= C' (P'^2 + p'q' + q'^2) \quad \text{where } C', p', q' \text{ are integer}$$

$$N^{2n} = C'^2 (P'^2 + p'q' + q'^2)^2$$

Thus there exists integers  $x, y, z$  such that

$$p'^2 + q'p' + q'^2 = x + y + z \quad \text{and}$$

$$(p'^2 + p'q' + q'^2)^2 = x^2 + y^2 + z^2$$

$$N^n = (C' (p'^2 + p'q' + q'^2) = c'(x + y + z) = (c'x) + (c'y) + (c'z) \text{ \&}$$

$$N^{2n} = (C'^2 (p'^2 + p'q' + q'^2)^2 = c'^2 (x^2 + y^2 + z^2) = (c'x)^2 + (c'y)^2 + (c'z)^2$$

The above analysis leads to the following result stated as

**Theorem 1 :** If the integer  $n > 0$  and  $N$  a natural number having at least one prime factor of the form  $6k+1$ , the diophantine equations.

$$(N^n) = x_1 + x_2 + x_3$$

$$(N^n)^2 = x_1^2 + x_2^2 + x_3^2$$

have always a solution.

**Result :1**

$p^2 + pq + q^2$  remains unaltered if we replace  $p$  by  $p+q$  and  $q$  by  $-q$  or if we replace  $p$  by  $p+q$  and  $q$  by  $-p$

$$(i.e) \quad p^2 + pq + q^2 = (p+q)^2 + (p+q)(-q) + (-q)^2$$

$$= (p+q)^2 + (p+q)(-p) + (-p)^2$$

For each of the above substitutions for  $p$  and  $q$  the 3 numbers  $-pq, p^2 + pq, q^2 + pq$  are only permutations leading to the same identities (6).

**Result : 2**

If we interchange either  $a, b$  or  $c, d$  (but not both simultaneously) in the form (2) we get a new set of values of  $A$  &  $B$  (which is the same for either interchange) through the product of LHS of (2) does not change.

Example :

$$N = 247 = (13) \quad (19) = (1^2 + (1)(3) + 3^2) \\ (2^2 + (2)(3) + 3^2) = 17^2 - \\ (17)(14) + 14^2 = 3^2 + 3(14) + 14^2 \rightarrow (i1) \\ = (3^2 + 3(1) + 1^2)(2^2 + 2(3) + 3^2) \\ = 11^2 - 11(18) + 18^2 \\ = 7^2 + 7(11) + 11^2 \rightarrow (ii1)$$

$$\text{and we have } 247 = (-42) + 51 + 238 = (-77) + 126 + 198$$

$$247^2 = (-42)^2 + 51^2 + 238^2 = (-77)^2 + 126^2 + 198^2$$

$$247^2 = (247)(247) \\ = (3^2 + 3(14) + 14^2)^2 \\ = (7^2 + 7(11) + 11^2)^2 \\ = (3^2 + 3(14) + 14^2)(7^2 + 7(11) + 11^2) \\ = (14^2 + 14(3) + 3^2)(7^2 + 7(11) + 11^2) \\ = (187^2 + 187(93) + 93^2) \rightarrow (i2) \\ = 72^2 + 72(203) + 203^2 \rightarrow (ii2) \\ = 65^2 + 65(208) + 208^2 \\ \rightarrow (iii2) \\ = 133^2 + 133(152) + 152^2 \\ \rightarrow (iv2)$$

and with (i2) we have

$$247^2 = (-17391) + 52360 + 26040 +$$

$$247^2 = (-17391)^2 + 52360^2 + 26040^2 \text{ (from identities (6) and so for (ii2), (iii2), (iv2)}$$

Again taking (ii1) and (i2) we have

$$247^3 = 247^2 (247) \\ = (187^2 + 187(93) + 93^2)(7^2 + 7(11) + 11^2) \\ = (2983^2 - 2983(4389) + 4389^2) \\ = 1406^2 + 1406(2983) + 2983^2 \\ \rightarrow (i3) \\ = 286^2 + 286(3731) + 3731^2 \\ \rightarrow (ii3)$$

and with (i3) we have

$$247^3 = (-4194098) + 6170934 + 13092387$$

$$247^6 = (-4194098)^2 + 6170934^2 + 13092387^2 \text{ etc.,}$$

The identities (6) can be written as

$$(p^2 + pq + q^2) + (p^2 + pq + q^2) = p^2 + q^2 + (p+q)^2 \rightarrow (7)$$

$$(p^2 + pq + q^2)^2 + (p^2 + pq + q^2)^2 = p^4 + q^4 + (p+q)^4 \rightarrow (8)$$

Hence we have

**Theorem :2**

If the integer  $n > 0$  &  $N$  a natural number having at least one set of  $U_1, U_2, U_3$  where  $U_1 + U_2 = U_3$  which satisfy the Diophantine equations.

$$N^n + N^n = U_1^2 + U_2^2 + U_3^2$$

$$N^{2n} + N^{2n} = U_1^4 + U_2^4 + U_3^4$$

**Example :**

With (i1) & (ii2) of above and previous we have

$$247 + 247 = 3^2 + 14^2 + 17^2 = 7^2 + 11^2 + 18^2$$

$$(247 + 247)^2 = 3^4 + 14^4 + 17^4 = 7^4 + 11^4 + 18^4$$

Now we consider the solutions of Diophantine equations for the value of  $h=3$  in the expansions

$$N^{nh} = x^h + y^h + z^h \text{ where } N = p^2 + pq + q^2$$

Proceeding as above we have the values

$$x = -pq, y + z = (p+q)^2, y^2 + z^2 = (p^2 + q^2)(p+q)^2$$

$$\text{consider } y^3 + z^3 = (y+z)[y^2 - yz + z^2]$$

$$= (p+q)^2 [(p^2 + q^2)(p+q)^2 - yz] \rightarrow (9)$$

$$y^3 + z^3 = N^3 - x^3 \text{ from our assumption}$$

$$= (N-x)(N^2 + Nx + x^2)$$

$$= (p+q)^2 [p^2 + pq + q^2](p^2 + q^2) + p^2 q^2 \rightarrow (10)$$



From equation (9) & (10)

$$yz = pq(p^2 + q^2) - p^2q^2$$

Now consider the identity

$$(y-z)^2 = (y+z)^2 - 4yz$$

$$= (p+q)^4 - 4pq(p^2 + q^2) + 4p^2q^2$$

$$= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4 - 4p^3q -$$

$$4pq^3 + 4p^2q^2$$

$$= p^4 + 10p^2q^2 + q^4$$

$$(y-z) = \sqrt{p^4 + 10p^2q^2 + q^4}$$

$$\text{Also } y+z = (p+q)^2$$

Solving the above two equations we get the irrational solutions of y & z.

### A PROOF OF FERMAT'S LAST THEOREM

Fermat's last theorem is based under the condition  $x^m + y^m = z^m$  which is impossible for natural numbers where m is greater than 2. (i.e) From the conditions  $x^m + y^m \neq z^m$  where  $x, y, z, m \in \mathbb{N}$  and  $m > 2$ .

But in the following three definite relations :

$$(i) \quad x^2 + y^2 = z^2 \text{ (Pythagorean triplet)}$$

$$(ii) \quad x^2 + y^2 < z^2$$

$$(iii) \quad x^2 + y^2 > z^2 \text{ (Inequality)}$$

(A) Case (1)

$$x^2 + y^2 = z^2, \quad x < z, y < z \quad \rightarrow (1)$$

Multiply equations (1) on both sides by  $x^{m-2}$

$$= x^{m-2}x^2 + y^2x^{m-2} = x^{m-2}z^2 \quad \rightarrow (2)$$

Since from the assumption  $x < y < z$  in (1)

$$\text{We get } (y^{m-2} - x^{m-2}) < (z^{m-2} - x^{m-2})$$

$$\text{Also } \Rightarrow y^2(y^{m-2} - x^{m-2}) < z^2(z^{m-2} - x^{m-2}) \quad (\text{since } y < z) \rightarrow (3)$$

Adding (2) & (3) we get the result  $x^m + y^m < z^m$

$$\Rightarrow x^m + y^m \neq z^m$$

B) It may be proved also, multiplying (1) by

$$(xyz)^{m-2}$$

$$\therefore x^2 + y^2 = z^2 \Rightarrow x^2(xyz)^{m-2} + y^2(xyz)^{m-2} = z^2(xyz)^{m-2}$$

$$x^m(yz)^{m-2} + y^m(xz)^{m-2} = z^m(xz)^{m-2}$$

Since from the assumption

$$x < y < z, (xz)^{m-2} < (zx)^{m-2} < (yz)^{m-2}$$

Now  $x^m + y^m = z^m$  is possible,

$$(xy)^{m-2} = (zx)^{m-2} = (yz)^{m-2}$$

But it is impossible to our assumptions  $x < y < z$ .

$$\text{So } x^m + y^m \neq z^m$$

$$(C) \text{ Case 2 : } x^2 + y^2 < z^2 \text{ where } x < y < z \rightarrow (4)$$

Multiply equations (4) on both sides by  $x^{m-2}$

$$x^{m-2}x^2 + y^2x^{m-2} < x^{m-2}z^2 \quad \rightarrow (5)$$

Since from the assumption  $x < y < z$

$$\text{We get } (y^{m-2} - x^{m-2}) < (z^{m-2} - x^{m-2})$$

$$\text{Also } \Rightarrow y^2(y^{m-2} - x^{m-2}) < z^2(z^{m-2} - x^{m-2}) \quad (\text{since } y < z) \rightarrow (6)$$

Adding (5) & (6) we get  $x^m + y^m < z^m$

$$\Rightarrow x^m + y^m \neq z^m$$

(D) It also proved as from inequality  $x < y < z$  for our assumption  $x^2 + y^2 < z^2$

$$\Rightarrow x^2 < y^2 < z^2$$

$$\Rightarrow z^2 - (x^2 + y^2) < z^m - (x^m + y^m) \quad (\text{since } m > 2)$$

$$\Rightarrow x^m + y^m < z^m$$

(E) case : 3

$$x^2 + y^2 > z^2 \text{ where } (i) x > z \text{ \& } y > z$$

$$(ii) z > x \text{ \& } y > z$$

$$(iii) x > z \text{ \& } z > y$$

The above inequality is the back bone for this problem. We have to prove that for our above assumption in another way.

F (i.e) consider  $x^2 + y^2 > z^2$   $x < z$  &  $y < z$

Let it be assumed to be such that  $x^2 + y^2 > z^2$  alters to  $\Rightarrow x^3 + y^3 = z^3$  for  $m=3$

If  $x_1, y_1, z_1$  are of a phythagorean triplet then

$$x_1^2 + y_1^2 = z_1^2$$

Multiplying the above result by a on both sides

$$ax_1^2 + ay_1^2 = az_1^2$$

It may be proved as in a different manner

$$V_1(\text{cuboid}_1) + V_2(\text{Cuboid}_2) = V_3(\text{cuboid}_3) \text{ of}$$

$$\text{Square base } x_1 \quad y_1$$

$$z_1$$

$$\text{Height } a \quad a \quad a$$

$$\text{Now } x^3 + y^3 = z^3$$

$$(i.e) \quad V_1(\text{cube}_1) + V_2(\text{Cube}_2) = V_3(\text{cube}_3) \text{ of}$$

$$\text{Square base } x \quad y \quad z$$

$$\text{Height } x \quad y \quad z$$

As cube may be steaped as cuboid we have

$$V_1(\text{cube}_1) = V_1(\text{cuboid}_1) \text{ and so on. } \therefore$$

$$x^3 = ax_1^2, y^3 = ay_1^2, z^3 = az_1^2$$

$$\therefore \text{ Certainly } (x^3, y^3, z^3) = d \text{ and } a \mid \forall x^3, y^3, z^3$$

$$\therefore x^3 / a = x_1^2, y^3 / a = y_1^2, z^3 / a = z_1^2 \text{ such that}$$

$$x_1^2 + y_1^2 = z_1^2 \text{ and}$$



$$x^3 / x_1^2 = y^3 / y_1^2 = z^3 / z_1^2$$

But this impossible because

$$x^2 / x_1^2 = y^2 / y_1^2 = z / z_1^2 \text{ is impossible}$$

In this way our supposition is false and

$$x^3 + y^3 \neq z^3 \text{ as well as}$$

$$x^m + y^m \neq z^m$$

(G) It may be proved as under too  $x < y < z$

$$\text{For } m=3; \text{ (i) } 2y^2 < 3(z^2 - x^2) \Rightarrow x^3 + y^3 < z^3$$

$$\text{(ii) } 2y^2 > 3(z^2 - x^2) \Rightarrow x^3 + y^3 > z^3$$

In general

$$\text{(i) } 2y^{m-1} < 3(z^{m-1} - x^{m-1}) \Rightarrow x^m + y^m < z^m$$

$$\text{(ii) } 2y^{m-1} > 3(z^{m-1} - x^{m-1}) \Rightarrow x^m + y^m > z^m$$

$$\therefore x^m + y^m \neq z^m$$

## CONCLUSION

*So far we have seen how continued fractions solves the Diophantine Equations in the Applied Mathematics. Thus Applied Mathematics without Pure Mathematics has no root and Pure Mathematics without Applied Mathematics leaves no fruit.*

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